Additional Separated-Variable Solutions of the Biharmonic Equation in Polar Coordinates

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*From the biharmonic equation of the plane problem in the polar coordinate system and taking into account the variable-separable form of the partial solutions, a homogeneous ordinary differential equation (ODE) of the fourth order is deduced. Our study is based on the investigation of the behavior of the coefficients of the above fourth order ODE, which are functions of the radial coordinate r. According to the proposed investigation additional terms, $\bar{\varphi}_{-m}(r,\theta)$ $(1\leq m\leq n)$ other than the usually tabulated in the Michell solution (1899, "On the Direct Determination of Stress in an Elastic Solid, With Application to the Theory of Plates," Proc. Lond. Math. Soc., **31**, pp. 100–124) are found. Finally the stress and the displacement fields due to each one additional term of $\bar{\varphi}_{-m}(r,\theta)$ are determined.* [DOI: 10.1115/1.3197157]

1 Introduction

Joints of different materials have many applications in structural engineering or microelectronics. Due to the different elastic properties and to different thermal expansion coefficients of the joined materials, stresses develop under mechanical loadings and/or after a change in temperature. In order to study the behavior of such joints an accurate calculation of the developed stresses under the applied loading conditions and an assessment of the stresses are required. The main problem in these joints consists in the fact that the stresses are especially high at the free edge of the interface (open wedge) or at an internal corner.

In the case of a two-dimensional situation, the in plane stress fields can be calculated by using an Airy stress function [1–4]. The stress tensor near the free edge of the interface is given by [5–16]

$$\sigma_{ij}=\frac{K_L}{(r/L)^{\omega}}f_{ij}(\theta)+\sigma_0 g_{ij}(\theta),\quad i,j=r,\theta \tag{1}$$

where r,θ are the polar coordinates, K_L is the stress intensity factor [9,11,12] related to a characteristic size parameter L and depending on the applied loadings at the joint, ω is a stress singularity $(\omega>0)$, f_{ij} and g_{ij} are functions of the angle θ depending on the angles of the joint, and σ_0 is an elastic constant depending on the applied loading.

For some combinations of wedge angles and elastic constants of the material, the stress singularity ω is complex, $\omega=b_1+ib_2$ [9,11,12]. Hence Eq. (1) may be written as

$$\sigma_{ij}=\frac{K_L}{(r/L)^{b_1}}(\cos b_2(\ln(r/L))f^c_{ij}(\theta)+\sin(b_2\ln(r/L))f^s_{ij}(\theta))$$
$$+\sigma_0 g_{ij}(\theta),\quad i,j=r,\theta \tag{2}$$

where f^c_{ij} and f^s_{ij} are functions of the angle θ depending on the angle of the joint.

In Eq. (2) the terms of the forms $\cos(b_2\ln(r/L))$ and $\sin(b_2\ln(r/L))$ appear. It is observed that the behavior of such terms has a sinusoidal form with a period increased by a geometric progression, which is given by

$$r_{\ell+1}=r_\ell \ell^{\pi/b_2},\quad \ell=0,1,2,\ldots \tag{3}$$

It should be noted that the above relation is self-similar. A lot of investigators [13–19] have studied two-dimensional elasticity problems through Michell's polar analysis resulting in sinusoidal expressions of form (3). Especially in the study of Leung and Zheng [18] for the closed form distribution in two-dimensional elasticity for all boundary conditions, it is observed that in the case of nonzero eigensolution terms $(\mu\neq 0)$ of the forms $\sin\lambda_i\mu z$ and $\cos\lambda_i\mu z$ (where $\lambda_i (i=1,2)$ are real numbers and z is the height of the beam) are involved in the stress and displacement field expressions when the characteristic polynomial of the governing differential equation has complex roots (Ref. [18], relation (30c)). Piva and Viola, examining the interfacial normal and shear stresses near the crack tip between dissimilar media under biaxial loads (Ref. [7], relations (5.2)–(5.5) and Appendix I), found expressions of the stress field containing sinusoidal terms of the forms $\cos(\varepsilon\ln(r/\ell))$, $\cos(\varepsilon\ln(r/\ell)-\delta)$, $\sin(\varepsilon\ln(r/\ell))$, and $\sin(\varepsilon\ln(r/\ell)-\delta)$ (where ε and δ are the parameters due to the oscillatory character of the stress field close to the crack tip [7] and ℓ is the length of the crack). Hutchinson and Suo [8] in the study of mixed mode cracking in layered materials, as well as Carlsson and Prasad [10] in their work of interfacial fracture of sandwich beams, expressed the associated crack flank displacements at a distance behind the crack's tip (Ref. [8], relation (2.27) and Ref. [10], relation (5)), in terms of $r^{i\varepsilon}=\cos(\varepsilon\ln r)+i\sin(\varepsilon\ln r)$. Thus the reason why separated-variable solutions are useful for the "edge" problem is that if the function of θ is made to go to zero at a fixed value, then the biharmonic function will also go to zero there for all r. But because of the oscillatory character of the terms $\cos(\varepsilon\ln r)$, $\sin(\varepsilon\ln r)$, $\cos(\varepsilon\ln(r/l)-\delta)$, and $\sin(\varepsilon\ln(r/l)-\delta)$, which appeared in these problems, the classical Michell's solution in polar coordinates cannot be applied.

In our study by investigating the Michell solution and by taking into account the variable-separable form of the partial solutions, additional variable-separable terms of the form $f(\theta)\cos(m\ln r)$ or $f(\theta)\sin(m\ln r)$ $(m=1,2,\ldots)$, other than the usually tabulated, are deduced. The importance of this study is that the additional separated-variable solutions of the biharmonic equation in polar coordinates may confront elasticity problems where the order of singularity appears in complex from.

At first the general solution of the biharmonic problem, which leads to well known results and is included in our study for the sake of completeness, is investigated. From the above analysis the constants b_1 and b_2 are determined. By examining the behavior of

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b_1 and in the case that $b_1 < 1$ (Sec. 2.2.2) the additional variable-separable terms result. The stress and displacement fields based on these terms are determined and tabulated. Even if a complex solution of the form $z^{\delta}=(re^{i\theta})^{a+i\beta}$ is considered, a class of additional variable-separable terms similar to the above mentioned occurs. The terms resulted from the proposed method concern the real form solutions of the investigated biharmonic equation. Hence with the new proposed terms, Michell's solution in polar coordinates is improved in order to be also applicable in the case of complex singularities in plane elasticity problems.

Finally an application is made where the coincidence under certain circumstances between Williams' stress function [20] and the proposed additional separated-variable solutions of the biharmonic equation in polar coordinates is proved.

2 General Solution

Let us seek in the polar coordinate system (r,θ), the solution of the biharmonic equation for the plane problem [1–3]

$$\nabla^4\Phi(r,\theta)=0 \Leftrightarrow \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\Phi(r,\theta)=0 \tag{4}$$

where

$$\Phi(r,\theta)=\sum_{m=o}^{n}\varphi_m(r,\theta), \quad n\in\mathbb{N} \tag{5}$$

and $\varphi_m(r,\theta)$ the partial solutions of Eq. (4), which may be assumed to be given in a variable-separable form by

$$\varphi_m(r,\theta)=h_m(r)g_m(\theta), \quad m=0,1,2,\ldots \tag{6}$$

From the fulfillment of the biharmonic equation (4) in relation to Eqs. (5) and (6) is obtained

$$g_m^{(iv)}(\theta)+2\Lambda_m(r)g_m''(\theta)+M_m(r)g_m(\theta)=0 \tag{7}$$

where

$$\Lambda_m(r)=r^2\frac{h_m''(r)}{h_m(r)}-r\frac{h_m'(r)}{h_m(r)}+2, \quad h_m(r)\neq 0$$

$$M_m(r)=r^4\frac{h_m^{(iv)}(r)}{h_m(r)}+2r^3\frac{h_m'''(r)}{h_m(r)}-r^2\frac{h_m''(r)}{h_m(r)}+r\frac{h_m'(r)}{h_m(r)}, \quad h_m(r)\neq 0 \tag{8}$$

and $h_m'=dh_m/dr$.

From a partial differentiation of Eq. (7) in terms of r, we get

$$2\Lambda_m'(r)g_m''(\theta)+M_m'(r)g_m(\theta)=0, \quad m=0,1,2,\ldots \tag{9}$$

In order to facilitate the investigation of the biharmonic equation (4) and without losing the generality of the solution, the assumption that every constant appeared in the proposed analysis is considered as a physical number is made. It is obvious that the following analysis may also be done by considering the appeared constants as real numbers.

In order to determine the roots of Eq. (9), Λ_m' $(m=0,1,2,\ldots)$ must be verified. Λ_m' are determined in the following study.

2.1 First Case $\Lambda_m'(r)\neq 0$. Equation (9) may be written as

$$\frac{g_m''(\theta)}{g_m(\theta)}=-\frac{M_m'(r)}{2\Lambda_m'(r)}=b=\text{const} \tag{10}$$

From Eq. (10) and assuming that from all the possible constants a constant of the form $b=-m^2$ $(m\in\mathbb{N})$ is chosen, we finally have

$$g_m''(\theta)+m^2g_m(\theta)=0, \quad m\in\mathbb{N} \tag{11}$$

The general solution of the homogeneous differential equation (11) is (see Appendix)

$$g_m(\theta)=\sum_{\ell=1}^{\mu=1}\sum_{\xi_\ell=1}^{\xi_1=1}(A_{\ell\xi_\ell}\cos\zeta_\ell\theta+B_{\ell\xi_\ell}\sin\zeta_\ell\theta)\theta^{\xi_\ell-1}e^{r_\ell\theta}$$
$$=A_{11}\cos(m\theta)+B_{11}\sin(m\theta) \tag{12}$$

From Eq. (11) the following is obtained:

$$g_m''(\theta)=-m^2g_m(\theta), \quad g^{(iv)}(\theta)=-m^2g_m''(\theta)=m^4g_m(\theta) \tag{13}$$

Equation (7), because of relations (13) and (8), finally becomes

$$\ddddot{h}_m-4\dddot{h}_m+(4-2m^2)\ddot{h}_m+4m^2\dot{h}_m+m^2(m^2-4)h_m=0;$$
$$r=e^t, \quad h_m(r)=h_m(e^t), \quad m\in\mathbb{N} \tag{14}$$

where $\dot{h}_m=dh_m/dt$, and with characteristic roots,

$$\rho_1=m, \quad \rho_2=m+2, \quad \rho_3=-m, \quad \rho_4=-m+2; \quad m=1,2,\ldots \tag{15}$$

2.1.1 Investigation of the Characteristic Roots of Equation (14). The characteristic roots of differential Eq. (14) will be determined in the following three cases:

(i) $m=0$.

The differential equation (11) has a solution of the form $g_0(\theta)=K_{01}\theta+K_{02}$, where K_{01} and K_{02} are arbitrary constants while from relation (15) the characteristic roots become

$$\rho_1=\rho_3=0, \quad \rho_2=\rho_4=2 \tag{16}$$

The general solution of Eq. (14) is finally given by (see Appendix)

$$h_0(r)=C_{11}+C_{12}\ln r+C_{21}r^2+C_{22}r^2\ln r \tag{17}$$

Hence the partial solution of Eq. (4) for $\Lambda_0'(r)\neq 0$, and because of relations (11) and (17), becomes

$$\varphi_0(r,\theta)=(\Gamma_{01}r^2+\Gamma_{02}r^2\ln r+\Gamma_{03}+\Gamma_{04}\ln r)+(\Gamma_{01}'r^2+\Gamma_{02}'r^2\ln r$$
$$+\Gamma_{03}'+\Gamma_{04}'\ln r)\theta \tag{18}$$

(ii) $m=1$.

From relation (15) the characteristic roots of the differential equation (14) become

$$\rho_1=\rho_4=1, \quad \rho_2=3, \quad \rho_3=-1 \tag{19}$$

The general solution of Eq. (14) is (see Appendix)

$$h_1(r)=C_{11}r+C_{12}r\ln r+C_{21}r^3+C_{31}r^{-1} \tag{20}$$

Hence the partial solution, φ_1, of Eq. (4) for $\Lambda_1'(r)\neq 0$, because of relations (11) and (20), is given by

$$\varphi_1(r,\theta)=(\Gamma_{11}r^3+\Gamma_{12}r^{-1}+\Gamma_{13}r+\Gamma_{14}r\ln r)\cos\theta+(\Gamma_{11}'r^3+\Gamma_{12}'r^{-1}$$
$$+\Gamma_{13}'r+\Gamma_{14}'r\ln r)\sin\theta \tag{21}$$

(iii) $2\leq m\leq n$.

The general solution of Eq. (14) is (see Appendix)

$$h_m(r)=C_{11}r^m+C_{21}r^{m+2}+C_{31}r^{-m}+C_{41}r^{-m+2}, \quad r=e^t \tag{22}$$

Hence, the partial solution φ_m of Eq. (4) for $\Lambda_m'(r)\neq 0$, because of relations (11) and (22), is

$$\varphi_m(r,\theta)=(\Gamma_{m1}r^{m+2}+\Gamma_{m2}r^m+\Gamma_{m3}r^{-m+2}+\Gamma_{m4}r^{-m})\cos(m\theta)$$
$$+(\Gamma_{m1}'r^{m+2}+\Gamma_{m2}'r^m+\Gamma_{m3}'r^{-m+2}+\Gamma_{m4}'r^{-m})\sin(m\theta),$$
$$2\leq m\leq n \tag{23}$$

2.2 Second Case $\Lambda'_m(r)=0$.

From Eq. (9) in the case that $g_m(\theta)\neq 0$, $M'_m(r)=0$ is also obtained.

Thus we get

$$\Lambda_m(r)=b_1=\text{const.},\quad M_m(r)=b_2=\text{const} \tag{24}$$

The homogeneous differential equation (7) because of relation (24) is written as

$$g_m^{(iv)}(\theta)+2b_1 g''_m(\theta)+b_2 g_m(\theta)=0 \tag{25}$$

where b_1 and b_2 are arbitrary constants.

Hence the determination of the partial solutions of Eq. (4) is reduced to the solution of the system of homogeneous differential equations (24) and (25).

Relation (24) because of relation (8) results in the differential system

$$\ddot{h}_m-2\dot{h}_m+(2-b_1)h_m=0,\quad r=e^t \tag{26a}$$

$$\ddddot{h}_m-4\dddot{h}_m+4\ddot{h}_m-b_2 h_m=0,\quad r=e^t \tag{26b}$$

The characteristic polynomial of Eq. (26*a*) is

$$(\rho-1)^2=b_1-1 \tag{27}$$

According to the sign (b_1-1), we distinguish two cases.

2.2.1 $b_1-1\geq 0$. Assuming that $b_1-1=m^2$ $(m=0,1,2,\ldots,n)$, the roots of the characteristic polynomial (27) are $\rho_1=1+m$ and $\rho_2=1-m$.

2.2.1.1 $m=0$. The general solution of Eq. (26*a*) is

$$\bar{h}_0(r)=C_{11}r+C_{12}r\ln r,\quad r=e^t \tag{28}$$

In order that Eq. (28) be the solution of system (24), it must satisfy Eq. (26*b*). Thus the following is obtained:

$$(1-b_2)C_{11}e^t+(1-b_2)C_{12}te^t=(1-b_2)\bar{h}_0=0 \tag{29}$$

Equation (29) holds only in the case that $b_2=1$.

In this case, the general solution of Eq. (26*b*) finally is

$$\bar{\bar{h}}_0(r)=\bar{h}_0(r)+C_{21}r^{1+\sqrt{2}}+C_{31}r^{1-\sqrt{2}},\quad r=e^t \tag{30}$$

In order that $\bar{\bar{h}}_0(r)$ satisfies Eq. (26*a*), $C_{21}=C_{31}=0$ must be valid and from relation (30) results

$$\bar{\bar{h}}_0(r)=\bar{h}_0(r) \tag{31}$$

Thus in the case that $b_1=b_2=1$ $(m=0)$, the solution of the differential system (24) is given by relation (28), and the homogeneous differential equation (25) is written as

$$g_0^{(iv)}(\theta)+2g''_0(\theta)+g_0(\theta)=0 \tag{32}$$

The general solution of the homogeneous differential equation (32) (see Appendix) is

$$\bar{g}_0(\theta)=\sum_{\ell=1}^{1}\sum_{\xi_\ell=1}^{2}(A_{\ell\xi_\ell}\cos\zeta_\ell\theta+B_{\ell\xi_\ell}\sin\zeta_\ell\theta)\theta^{\xi_\ell-1}e^{r_\ell\theta}$$

$$=A_{11}\cos\theta+B_{11}\sin\theta+A_{12}\theta\cos\theta+B_{12}\theta\sin\theta \tag{33}$$

Thus, Eq. (6), because of relations (28) and (33), and eliminating those terms that have already been included in the partial solution $\varphi_1(r,\theta)$ (Eq. (21)), becomes

$$\bar{\varphi}_0(r,\theta)=(\bar{\Gamma}_{01}r+\bar{\Gamma}_{02}r\ln r)\theta\cos\theta+(\bar{\Gamma}'_{01}r+\bar{\Gamma}'_{02}r\ln r)\theta\sin\theta \tag{34}$$

2.2.1.2 $m>0$ *and* $b_1=1+m^2>1$. In this case the general solution of Eq. (26*a*) becomes

$$\bar{h}_m(r)=C_{11}r^{1+m}+C_{21}r^{1-m},\quad r=e^t \tag{35}$$

In order that relation (28) be the solution of the differential system (24), Eq. (26*b*) must be satisfied. Hence, we finally have

$$[(1+m)^2(m-1)^2-b_2]C_{11}e^{(1+m)t}+[(1+m)^2(m-1)^2-b_2]C_{21}e^{(1-m)t}=0 \tag{36}$$

In order that Eq. (36) is valid for every t, $b_2=(1+m)^2(1-m)^2$ must hold.

Thus the characteristic polynomial of Eq. (26*b*) is

$$(\rho-1-m)(\rho-1+m)[(\rho-1)^2+m^2-2]=0$$

$$0<m(=1,2,\ldots,n)\leq n \tag{37}$$

We have to investigate the following two cases.

- Let $m=1$ and consequently $b_1=2$ and $b_2=0$.

Because of Eqs. (37) and (35), the general solution of Eq. (26*b*) finally becomes

$$\bar{\bar{h}}_1(r)=\bar{h}_1(r)+C'_{12}\ln r+C'_{22}r^2\ln r \tag{38}$$

In order that $\bar{\bar{h}}_1(r)$ satisfies Eq. (26*a*) for every $t(r=e^t)$, it must hold that $C'_{12}=C'_{22}=0$, and from relation (38) we have $\bar{\bar{h}}_1(r)=\bar{h}_1(r)$.

Thus in the case that $b_1=2$ and $b_2=0$ $(m=1)$, the solution of the differential system (24) is given by

$$\bar{h}_1(r)=C_{11}r^2+C_{21} \tag{39}$$

and the homogeneous differential equation (25) becomes

$$g_1^{(iv)}(\theta)+4g''_1(\theta)=0 \tag{40}$$

The general solution of Eq. (40) (see Appendix) is

$$\bar{g}_1(\theta)=\bar{C}_{11}+\bar{C}_{12}\theta+A_{11}\cos 2\theta+B_{11}\sin 2\theta \tag{41}$$

Relation (6) because of relations (39) and (41) becomes

$$\bar{\varphi}_1(r,\theta)=(C_{11}r^2+C_{21})(\bar{C}_{11}+\bar{C}_{12}\theta)+(C_{11}r^2+C_{21})(A_{11}\cos 2\theta+B_{11}\sin 2\theta) \tag{42}$$

It can be easily seen that the partial solution $\bar{\varphi}_1(r,\theta)$ is already included in the partial solutions $\varphi_0(r,\theta)$ and $\varphi_2(r,\theta)$ (relations (18) and (23)).

- Let $2\leq m\leq n$; $m=2,3,\ldots,n$, $b_1=1+m^2$, and $b_2=(1-m^2)^2$.

The general solution of Eq. (26*b*) because of relations (37) and (35) becomes

$$\bar{\bar{h}}_m(r)=\bar{h}_m(r)+[A_{11}\cos((\sqrt{m^2-2})\ln r)+B_{11}\sin((\sqrt{m^2-2})\ln r)]r,\quad r=e^t \tag{43}$$

In order that relation (43) satisfies Eq. (26*a*) for every t, it must hold that $A_{11}=B_{11}=0$.

Thus from relation (43), it yields $\bar{\bar{h}}_m(r)=\bar{h}_m(r)$. Hence in the case that $b_1=1+m^2$, $b_2=(1-m^2)^2$ $(2\leq m\leq n,\ m=2,3,\ldots,n)$, the solution of the differential system (24) is given by relation (35), namely,

$$\bar{h}_m(r)=C_{11}r^{1+m}+C_{21}r^{1-m}$$

and the homogeneous differential equation (25) is written as

$$g_m^{(iv)}(\theta)+2(1+m^2)g''_m(\theta)+(1-m^2)^2 g_m(\theta)=0 \tag{44}$$

with general solution

$$\bar{g}_m(\theta) = A_{11}\cos[(1+m)\theta] + B_{11}\sin[(1+m)\theta] + A_{21}\cos[(1-m)\theta] + B_{21}\sin[(1-m)\theta] \quad (45)$$

Thus Eq. (6) because of relations (35) and (45) becomes

$$\bar{\varphi}_m(r,\theta) = \bar{h}_m(r)\bar{g}_m(\theta) = (C_{11}r^{1+m} + C_{21}r^{1-m})\{A_{11}\cos[(1+m)\theta] + B_{11}\sin[(1+m)\theta]\} + (C_{11}r^{1+m} + C_{21}r^{1-m})\{A_{21}\cos[(1-m)\theta] + B_{21}\sin[(1-m)\theta]\} \quad (46)$$

It can be easily seen that the partial solution $\bar{\varphi}_m(r,\theta)$ is accomplished by the partial solutions $\varphi_1(r,\theta)$ and $\varphi_m(r,\theta)$ (relations (21) and (23)).

2.2.2 $b_1 - 1 < 0$. Assuming that $b_1 - 1 = -m^2$ $(m = 1, 2, \ldots, n)$, the characteristic polynomial (27) of Eq. (26*a*) becomes

$$(\rho - 1 - im)(\rho - 1 + im) = 0 \quad (47)$$

where $i = \sqrt{-1}$, and it possesses a single pair of complex conjugate roots.

The general solution of Eq. (26*a*) becomes

$$\bar{h}_{-m}(r) = [A_{11}\cos(m \ln r) + B_{11}\sin(m \ln r)]r, \quad r = e^t \quad (48)$$

In order that relation (48) be the solution of the differential system (24), it must also satisfy Eq. (26*b*). Thus we finally get

$$[(m^2+1)^2 - b_2](A_{11}e^t \cos mt + B_{11}e^t \sin mt) = [(m^2+1)^2 - b_2]\bar{h}_{-m} = 0$$

In order that the above equation be valid for every t, we must have

$$b_2 = (1+m^2)^2 \quad (49)$$

Hence, the general solution of Eq. (26*b*), taking into account relation (48), becomes

$$\bar{\bar{h}}_{-m}(r) = \bar{h}_{-m}(r) + C_{11}e^{(1+\sqrt{2+m^2})t} + C_{21}e^{(1-\sqrt{2+m^2})t} \quad (50)$$

From Eqs. (26*a*) and (50), it follows that

$$2(1+m^2)(C_{11}e^{(1+\sqrt{2+m^2})t} + C_{21}e^{(1-\sqrt{2+m^2})t}) = 0, \quad \forall\, t \in \mathbb{R}$$

So long as the above equation holds for every $t \in \mathbb{R}$, we finally have $C_{11} = C_{21} = 0$, and from relation (50) $\bar{\bar{h}}_{-m}(r) = \bar{h}_{-m}(r)$.

Thus the solution of the differential Eq. (26*a*), in the case that $b_1 = 1 - m^2$ and $b_2 = (1+m^2)^2$ $(1 \le m \le n)$, is given by relation (48). The homogeneous differential equation (25) is written as

$$g_{-m}^{(iv)}(\theta) + 2(1-m^2)g''_{-m}(\theta) + (1+m^2)^2 g_{-m}(\theta) = 0 \quad (51)$$

The general solution of Eq. (51) is

$$\bar{g}_{-m}(\theta) = \sum_{\ell=1}^{2}\sum_{\xi_\ell=1}^{1}(A'_{\ell\xi_\ell}\cos\zeta_\ell\theta + B'_{\ell\xi_\ell}\sin\zeta_\ell\theta)\theta^{\xi_\ell - 1}e^{r_\ell\theta} = (A'_{11}e^{m\theta} + A'_{21}e^{-m\theta})\cos\theta + (B'_{11}e^{m\theta} + B'_{21}e^{-m\theta})\sin\theta \quad (52)$$

Thus relation (6) because of relations (48) and (52) finally becomes

$$\bar{\varphi}_{-m}(r,\theta) = [(\bar{\gamma}_{m1}e^{m\theta} + \bar{\gamma}_{m2}e^{-m\theta})r\cos(m\ln r) + (\bar{\gamma}_{m3}e^{m\theta} + \bar{\gamma}_{m4}e^{-m\theta})r\sin(m\ln r)]\cos\theta + [(\bar{\gamma}_{m5}e^{m\theta} + \bar{\gamma}_{m6}e^{-m\theta})r\cos(m\ln r) + (\bar{\gamma}_{m7}e^{m\theta} + \bar{\gamma}_{m8}e^{-m\theta})r\sin(m\ln r)]\sin\theta \quad (53)$$

Hence relation (5), taking into consideration relations (18), (21), (23), (34), and (53), is written as

$$\Phi(r,\theta) = \varphi_0(r,\theta) + \varphi_1(r,\theta) + \sum_{m=2}^{n}\varphi_m(r,\theta) + \bar{\varphi}_0(r,\theta) + \sum_{m=1}^{n}\bar{\varphi}_{-m}(r,\theta) = M(r,\theta) \quad (54)$$

where

$$\varphi_0(r,\theta) = \Gamma_{01}r^2 + \Gamma_{02}r^2\ln r + \Gamma_{03} + \Gamma_{04}\ln r + (\Gamma'_{01}r^2 + \Gamma'_{02}r^2\ln r + \Gamma'_{03} + \Gamma'_{04}\ln r)\theta$$

$$\varphi_1(r,\theta) = (\Gamma_{11}r^3 + \Gamma_{12}r^{-1} + \Gamma_{13}r + \Gamma_{14}r\ln r)\cos\theta + (\Gamma'_{11}r^3 + \Gamma'_{12}r^{-1} + \Gamma'_{13}r + \Gamma'_{14}r\ln r)\sin\theta$$

$$\varphi_m(r,\theta) = (\Gamma_{m1}r^{m+2} + \Gamma_{m2}r^m + \Gamma_{m3}r^{-m+2} + \Gamma_{m4}r^{-m})\cos(m\theta) + (\Gamma'_{m1}r^{m+2} + \Gamma'_{m2}r^m + \Gamma'_{m3}r^{-m+2} + \Gamma'_{m4}r^{-m})\sin(m\theta)$$

$$2 \le m \le n$$

$$\bar{\varphi}_0(r,\theta) = (\bar{\Gamma}_{01}r + \bar{\Gamma}_{02}r\ln r)\theta\cos\theta + (\bar{\Gamma}'_{01}r + \bar{\Gamma}'_{02}r\ln r)\theta\sin\theta$$

$$\bar{\varphi}_{-m}(r,\theta) = [(\bar{\gamma}_{m1}e^{m\theta} + \bar{\gamma}_{m2}e^{-m\theta})r\cos(m\ln r) + (\bar{\gamma}_{m3}e^{m\theta} + \bar{\gamma}_{m4}e^{-m\theta})r\sin(m\ln r)]\cos\theta + [(\bar{\gamma}_{m5}e^{m\theta} + \bar{\gamma}_{m6}e^{-m\theta})r\cos(m\ln r) + (\bar{\gamma}_{m7}e^{m\theta} + \bar{\gamma}_{m8}e^{-m\theta})r\sin(m\ln r)]\sin\theta, \quad 1 \le m \le n \quad (55)$$

and

$$\Gamma_{ij}, \Gamma'_{ij}; \quad i = 0, 1, \ldots, m, \ldots, n, \quad j = 1, 2, 3, 4$$

$$\bar{\Gamma}_{01}, \quad \bar{\Gamma}_{02}, \quad \bar{\Gamma}'_{01}, \quad \bar{\Gamma}'_{02}$$

$$\bar{\gamma}_{ij}; \quad i = 1, 2, \ldots, m, \ldots, n, \quad j = 1, 2, \ldots, 8$$

arbitrary constants to be determined from the boundary conditions.

The coefficients Γ_{03}, Γ_{13}, and Γ'_{13} may be posed equal to zero because the corresponding terms of the stress function (54) do not cause a stress field.

Equation (54) constitutes the general Michell solution including the additional terms $\bar{\varphi}_{-m}(r,\theta)$ $(1 \le m \le n)$.

The stress and displacement fields due to each one term included in $\varphi_0(r,\theta)$, $\varphi_1(r,\theta)$, $\varphi_m(r,\theta)$, and $\bar{\varphi}_0(r,\theta)$ of relation (55) have already been determined in Ref. [4].

On the other hand the stress and displacement fields due to the terms $\bar{\varphi}_{-m}(r,\theta)$ $(1 \le m \le n)$ are given for the first time in Sec. 3, as the complementary solutions of Michell's problem in polar coordinates.

3 Stress and Displacement Fields Due to $\bar{\varphi}_{-m}(r,\theta)$

From relation (55), the terms $\bar{\varphi}_{-m}(r,\theta)$ may be written as follows:

$$\bar{\varphi}_{-m}(r,\theta) = \bar{\gamma}_{m1}\bar{\varphi}_1 + \bar{\gamma}_{m2}\bar{\varphi}_2 + \bar{\gamma}_{m3}\bar{\varphi}_3 + \bar{\gamma}_{m4}\bar{\varphi}_4 + \bar{\gamma}_{m5}\bar{\varphi}_5 + \bar{\gamma}_{m6}\bar{\varphi}_6 + \bar{\gamma}_{m7}\bar{\varphi}_7 + \bar{\gamma}_{m8}\bar{\varphi}_8 \quad (56)$$

where

$$\bar{\varphi}_1 = r\cos(m\ln r)e^{m\theta}\cos\theta, \quad \bar{\varphi}_2 = r\cos(m\ln r)e^{-m\theta}\cos\theta, \quad 1 \le m \le n$$

$$\bar{\varphi}_3 = r\sin(m\ln r)e^{m\theta}\cos\theta, \quad \bar{\varphi}_4 = r\sin(m\ln r)e^{-m\theta}\cos\theta, \quad 1 \le m \le n$$

$$\bar{\varphi}_5 = r\cos(m\ln r)e^{m\theta}\sin\theta, \quad \bar{\varphi}_6 = r\cos(m\ln r)e^{-m\theta}\sin\theta, \quad 1 \le m \le n$$

Table 1 The complementary stress fields

$\bar{\varphi}_i(r,\theta)$	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
$\bar{\varphi}_1=r\cos(m\ln r)e^{m\theta}\cos\theta$, $1\le m\le n$	$\frac{me^{m\theta}}{r}[(m\cos\theta-\sin\theta)\cos(m\ln r)$ $-\sin(\theta+m\ln r)]$	$\frac{me^{m\theta}}{r}(m\cos\theta-\sin\theta)$ $\times\sin(m\ln r)$	$-\frac{me^{m\theta}}{r}\cos\theta[m\cos(m\ln r)$ $+\sin(m\ln r)]$
$\bar{\varphi}_2=r\cos(m\ln r)e^{-m\theta}\cos\theta$, $1\le m\le n$	$\frac{me^{-m\theta}}{r}[(m\cos\theta+\sin\theta)\cos(m\ln r)$ $+\sin(\theta-m\ln r)]$	$-\frac{me^{-m\theta}}{r}(m\cos\theta+\sin\theta)$ $\times\sin(m\ln r)$	$-\frac{me^{-m\theta}}{r}\cos\theta[m\cos(m\ln r)$ $+\sin(m\ln r)]$
$\bar{\varphi}_3=r\sin(m\ln r)e^{m\theta}\cos\theta$, $1\le m\le n$	$\frac{me^{m\theta}}{r}[(m\cos\theta-\sin\theta)\sin(m\ln r)$ $+\cos(\theta+m\ln r)]$	$-\frac{me^{m\theta}}{r}(m\cos\theta-\sin\theta)$ $\times\cos(m\ln r)$	$-\frac{me^{m\theta}}{r}\cos\theta[m\sin(m\ln r)$ $-\cos(m\ln r)]$
$\bar{\varphi}_4=r\sin(m\ln r)e^{-m\theta}\cos\theta$, $1\le m\le n$	$\frac{me^{-m\theta}}{r}[(m\cos\theta+\sin\theta)\sin(m\ln r)$ $+\cos(\theta-m\ln r)]$	$\frac{me^{-m\theta}}{r}(m\cos\theta+\sin\theta)$ $\times\cos(m\ln r)$	$-\frac{me^{-m\theta}}{r}\cos\theta[m\sin(m\ln r)$ $-\cos(m\ln r)]$
$\bar{\varphi}_5=r\cos(m\ln r)e^{m\theta}\sin\theta$, $1\le m\le n$	$\frac{me^{m\theta}}{r}[(m\sin\theta+\cos\theta)\cos(m\ln r)$ $+\cos(\theta+m\ln r)]$	$\frac{me^{m\theta}}{r}(m\sin\theta+\cos\theta)$ $\times\sin(m\ln r)$	$-\frac{me^{m\theta}}{r}\sin\theta[m\cos(m\ln r)$ $+\sin(m\ln r)]$
$\bar{\varphi}_6=r\cos(m\ln r)e^{-m\theta}\sin\theta$, $1\le m\le n$	$-\frac{me^{-m\theta}}{r}[(\cos\theta-m\sin\theta)\cos(m\ln r)$ $+\cos(\theta-m\ln r)]$	$-\frac{me^{-m\theta}}{r}(m\sin\theta-\cos\theta)$ $\times\sin(m\ln r)$	$-\frac{me^{-m\theta}}{r}\sin\theta[m\cos(m\ln r)$ $+\sin(m\ln r)]$
$\bar{\varphi}_7=r\sin(m\ln r)e^{m\theta}\sin\theta$, $1\le m\le n$	$\frac{me^{m\theta}}{r}[(m\sin\theta+\cos\theta)\sin(m\ln r)$ $+\sin(\theta+m\ln r)]$	$-\frac{me^{m\theta}}{r}(m\sin\theta+\cos\theta)$ $\times\cos(m\ln r)$	$\frac{me^{m\theta}}{r}\sin\theta[\cos(m\ln r)$ $-m\sin(m\ln r)]$
$\bar{\varphi}_8=r\sin(m\ln r)e^{-m\theta}\sin\theta$, $1\le m\le n$	$\frac{me^{-m\theta}}{r}[(m\sin\theta-\cos\theta)\sin(m\ln r)$ $+\sin(\theta-m\ln r)]$	$-\frac{me^{-m\theta}}{r}(\cos\theta-m\sin\theta)$ $\times\cos(m\ln r)$	$\frac{me^{-m\theta}}{r}\sin\theta[\cos(m\ln r)$ $-m\sin(m\ln r)]$

$$\bar{\varphi}_7=r\sin(m\ln r)e^{m\theta}\sin\theta,\quad \bar{\varphi}_8=r\sin(m\ln r)e^{-m\theta}\sin\theta,\quad 1\le m\le n \tag{57}$$

The stress field caused by each one term of relation (56) is given by [2–4]

$$\sigma_{rr}=\frac{1}{r}\frac{\partial\bar{\varphi}_i}{\partial r}+\frac{1}{r^2}\frac{\partial^2\bar{\varphi}_i}{\partial\theta^2},\quad \sigma_{\theta\theta}=\frac{\partial^2\bar{\varphi}_i}{\partial r^2},\quad \sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\bar{\varphi}_i}{\partial\theta}\right)$$
$$i=1,2,\ldots,8 \tag{58}$$

The stress fields (58) are tabulated in Table 1.

Taking into consideration the strain-stress relations [2–4]

$$\varepsilon_{rr}=\frac{1}{8\mu}[(1+\kappa)\sigma_{rr}-(3-\kappa)\sigma_{\theta\theta}],$$

$$\varepsilon_{\theta\theta}=-\frac{1}{8\mu}[(3-\kappa)\sigma_{rr}-(1+\kappa)\sigma_{\theta\theta}],$$

$$\varepsilon_{r\theta}=\frac{1}{2\mu}\sigma_{r\theta} \tag{59}$$

where $\kappa=3-4\nu$ for plane strain and $\kappa=(3-\nu)/(1+\nu)$ for plane stress, ν being Poisson's ratio, μ being the shear modulus, the strain-displacement relations [2–4]

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r},\quad \varepsilon_{r\theta}=\frac{1}{2r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta+r\frac{\partial u_\theta}{\partial r}\right) \tag{60}$$

and the quantities

$$R=\int\int\sigma_{rr}drd\theta,\quad \Theta=\int\int\sigma_{\theta\theta}drd\theta \tag{61}$$

the displacement field is given finally by

$$u_r=\left(\frac{1+\kappa}{8\mu}\right)\frac{\partial R}{\partial\theta}-\left(\frac{3-\kappa}{8\mu}\right)\frac{\partial\Theta}{\partial\theta}+\frac{dQ(\theta)}{d\theta},$$

$$u_\theta=\frac{1+\kappa}{8\mu}\left(r\frac{\partial\Theta}{\partial r}-R\right)-\frac{3-\kappa}{8\mu}\left(r\frac{\partial R}{\partial r}-\Theta\right)+P(r)-Q(\theta) \tag{62}$$

In relation (62) the functions $P(r)$ and $Q(\theta)$ are determined from the boundary conditions of the problem.

The displacement fields (62) due to the terms (57) are tabulated in Table 2.

Table 2 The complementary displacement fields

$\bar{\varphi}_i(r,\theta)$	$16\mu u_r$	$4\mu u_\theta$
$\bar{\varphi}_1 = r\cos(m\ln r)e^{m\theta}\cos\theta$, $1\le m\le n$	$(1+\kappa)e^{m\theta}[-\cos(\theta-m\ln r)+3\cos(\theta+m\ln r) -m\sin(\theta-m\ln r) +m\sin(\theta+m\ln r)]-2(3-\kappa)e^{m\theta}\cos\theta[\cos(m\ln r) -m\sin(m\ln r)]+Q'(\theta)$	$e^{m\theta}\{-\cos(m\ln r)[2m\cos\theta+(-1+\kappa)\sin\theta] -(1+\kappa)\cos\theta\sin(m\ln r)\}+P(r)-Q(\theta)$
$\bar{\varphi}_2 = r\cos(m\ln r)e^{-m\theta}\cos\theta$, $1\le m\le n$	$(1+\kappa)e^{-m\theta}[3\cos(\theta-m\ln r)-\cos(\theta+m\ln r) -m\sin(\theta-m\ln r) +m\sin(\theta+m\ln r)]-2(3-\kappa)e^{-m\theta}\cos\theta[\cos(m\ln r) -m\sin(m\ln r)]+Q'(\theta)$	$e^{-m\theta}\{\cos(m\ln r)[2m\cos\theta-(-1+\kappa)\sin\theta] +(1+\kappa)\cos\theta\sin(m\ln r)\}+P(r)-Q(\theta)$
$\bar{\varphi}_3 = r\sin(m\ln r)e^{m\theta}\cos\theta$, $1\le m\le n$	$(1+\kappa)e^{m\theta}[-m\cos(\theta-m\ln r)-m\cos(\theta+m\ln r) +\sin(\theta-m\ln r) +3\sin(\theta+m\ln r)]-2(3-\kappa)e^{m\theta}\cos\theta[m\cos(m\ln r) +\sin(m\ln r)]+Q'(\theta)$	$e^{m\theta}\{(1+\kappa)\cos\theta\cos(m\ln r)-[2m\cos\theta +(-1+\kappa)\sin\theta]\sin(m\ln r)\}+P(r)-Q(\theta)$
$\bar{\varphi}_4 = r\sin(m\ln r)e^{-m\theta}\cos\theta$, $1\le m\le n$	$-(1+\kappa)e^{-m\theta}[m\cos(\theta-m\ln r)+m\cos(\theta+m\ln r) +3\sin(\theta-m\ln r) +\sin(\theta+m\ln r)]-2(3-\kappa)e^{-m\theta}\cos\theta[m\cos(m\ln r) +\sin(m\ln r)]+Q'(\theta)$	$e^{-m\theta}\{-(1+\kappa)\cos\theta\cos(m\ln r)+[2m\cos\theta +(1-\kappa)\sin\theta]\sin(m\ln r)\}+P(r)-Q(\theta)$
$\bar{\varphi}_5 = r\cos(m\ln r)e^{m\theta}\sin\theta$, $1\le m\le n$	$(1+\kappa)e^{m\theta}[m\cos(\theta-m\ln r)-m\cos(\theta+m\ln r) -\sin(\theta-m\ln r) +3\sin(\theta+m\ln r)]-2(3-\kappa)e^{m\theta}\sin\theta[\cos(m\ln r) -m\sin(m\ln r)]+Q'(\theta)$	$e^{m\theta}\{-\cos(m\ln r)[(1-\kappa)\cos\theta+2m\sin\theta] -(1+\kappa)\sin\theta\sin(m\ln r)\}+P(r)-Q(\theta)$
$\bar{\varphi}_6 = r\cos(m\ln r)e^{-m\theta}\sin\theta$, $1\le m\le n$	$-(1+\kappa)e^{-m\theta}[-m\cos(\theta-m\ln r)+m\cos(\theta+m\ln r) -3\sin(\theta-m\ln r) +\sin(\theta+m\ln r)]-2(3-\kappa)e^{-m\theta}\sin\theta[\cos(m\ln r) -m\sin(m\ln r)]+Q'(\theta)$	$e^{-m\theta}\{\cos(m\ln r)[(-1+\kappa)\cos\theta+2m\sin\theta] +(1+\kappa)\sin\theta\sin(m\ln r)\}+P(r)-Q(\theta)$
$\bar{\varphi}_7 = r\sin(m\ln r)e^{m\theta}\sin\theta$, $1\le m\le n$	$-(1+\kappa)e^{m\theta}[\cos(\theta-m\ln r)+3\cos(\theta+m\ln r) +m\sin(\theta-m\ln r) +m\sin(\theta+m\ln r)]-2(3-\kappa)e^{m\theta}\sin\theta[m\cos(m\ln r) +\sin(m\ln r)]+Q'(\theta)$	$e^{m\theta}\{(1+\kappa)\cos(m\ln r)\sin\theta+[(-1+\kappa)\cos\theta -2m\sin\theta]\sin(m\ln r)\}+P(r)-Q(\theta)$
$\bar{\varphi}_8 = r\sin(m\ln r)e^{-m\theta}\sin\theta$, $1\le m\le n$	$(1+\kappa)e^{-m\theta}[3\cos(\theta-m\ln r)+\cos(\theta+m\ln r) -m\sin(\theta-m\ln r) -m\sin(\theta+m\ln r)]-2(3-\kappa)e^{-m\theta}\sin\theta[m\cos(m\ln r) +\sin(m\ln r)]+Q'(\theta)$	$e^{-m\theta}\{-(1+\kappa)\cos(m\ln r)\sin\theta+[(-1+\kappa)\cos\theta +2m\sin\theta]\sin(m\ln r)\}+P(r)-Q(\theta)$

4 Application

In this application the coincidence under certain circumstances between Williams' stress function [20] and the proposed additional separated-variable solutions of the biharmonic equation in polar coordinates is proved.

The in-plane singular displacement and stress fields at the apex of a two-dimensional, n-material wedge or junction can be obtained using Williams' stress function expressed in polar coordinates (Ref. [13], relation (2a)) by

$$\Phi_k(r,\theta) = Qr^{(2-\omega)}g_k(\theta), \quad k=1,2,\ldots,n \tag{63}$$

where k designates the kth material, ω is the order of the stress singularity at the apex of the wedge [20], Q an arbitrary constant, and $g_k(\theta)$ is the unknown function to be determined. In the case that the n-material wedge alters into an angularly graded wedge [21], then the functions $g_k(\theta)$ are substituted by $g(\theta)$.

When the stress singularity ω is complex,

$$\omega = 1-(\beta+i\varepsilon) \tag{64}$$

then Williams' stress function (63) is generally expressed by a stress function of the form (Ref. [13], relation (7))

$$\Phi(r,\theta) = b_1\,\mathrm{Re}[Qr^{(2-\omega)}g(\theta)] + b_2\,\mathrm{Im}[Qr^{(2-\omega)}g(\theta)] \tag{65}$$

where $Q=Q_1+iQ_2$ and $Q_1,Q_2,b_1,b_2\in\mathbb{R}$.

Relation (65) is a linear combination of the real and imaginary parts of the complex stress function (63), which satisfy the biharmonic equation.

Considering that

$$r^{(2-\omega)} = r^{(1+\beta)}r^{i\varepsilon} = r^{(1+\beta)}[\cos(\varepsilon\ln r)+i\sin(\varepsilon\ln r)] \tag{66}$$

Williams' stress function (65) can be written as

$$\Phi(r,\theta) = r^{(1+\beta)}\cos(\varepsilon\ln r)[M_1\,\mathrm{Re}\,g(\theta) - M_2\,\mathrm{Im}\,g(\theta)] - r^{(1+\beta)}\sin(\varepsilon\ln r)[M_1\,\mathrm{Im}\,g(\theta) + M_2\,\mathrm{Re}\,g(\theta)] \tag{67}$$

where $M_1=b_1Q_1+b_2Q_2$ and $M_2=b_1Q_2-b_2Q_1$.

In the case that the complex angular function $g(\theta)$ becomes

$$g(\theta) = (Ae^{\varepsilon\theta}+Be^{-\varepsilon\theta})\cos\theta + (Ce^{\varepsilon\theta}+De^{-\varepsilon\theta})\sin\theta \tag{68}$$

where $A=A_1+iA_2$, $B=B_1+iB_2$, $C=C_1+iC_2$, and $D=D_1+iD_2$; and taking that $\beta=0$ and $\varepsilon=m$, the stress function (67) is finally written as

$$\Phi(r,\theta) = [(\bar{\gamma}_{m1}e^{m\theta}+\bar{\gamma}_{m2}e^{-m\theta})r\cos(m\ln r) + (\bar{\gamma}_{m3}e^{m\theta} + \bar{\gamma}_{m4}e^{-m\theta})r\sin(m\ln r)]\cos\theta + [(\bar{\gamma}_{m5}e^{m\theta} + \bar{\gamma}_{m6}e^{-m\theta})r\cos(m\ln r) + (\bar{\gamma}_{m7}e^{m\theta}$$

$$+ \bar{\gamma}_{m8}e^{-m\theta})r \sin(m \ln r)]\sin \theta \quad (69)$$

where

$$\bar{\gamma}_{m1} = M_1A_1 - M_2A_2, \quad \bar{\gamma}_{m2} = M_1B_1 - M_2B_2, \quad \bar{\gamma}_{m3} = -M_1A_2 - M_2A_1, \quad \bar{\gamma}_{m4} = -M_1B_2 - M_2B_1$$

$$\bar{\gamma}_{m5} = M_1C_1 - M_2C_2, \quad \bar{\gamma}_{m6} = M_1D_1 - M_2D_2, \quad \bar{\gamma}_{m7} = -M_1C_2 - M_2C_1, \quad \bar{\gamma}_{m8} = -M_1D_2 - M_2D_1 \quad (70)$$

From relations (69) and (53) occurs that

$$\Phi(r,\theta) = \bar{\varphi}_{-m}(r,\theta), \quad 1 \le m \le n \quad (71)$$

Hence from relation (71) results that Williams' stress function (67) coincides under certain circumstances with the additional separated-variable Michell's solution (53).

5 Conclusions and Discussion

The scope of this paper are the formulation and the tabulation of the complement solutions deriving from the investigation of the solution of the biharmonic equation in polar coordinates. In our study additional terms than those usually tabulated in Michell's solution are determined. These complement solutions appeared in the case that the complex stress and displacement fields are expanded into their real form [9,11,12]. Furthermore these complement solutions constitute additional terms in the tables proposed in Ref. [4] (Tables 8.1 and 9.1) for the stress and displacement fields. Hence the most general separated-variable solution of the biharmonic equation in polar coordinates results from our study.

The complementary stress and displacement terms are used in several applications of two-dimensional elasticity. In the stress and displacement field expressions at the neighbor of an interface crack between dissimilar media [7,8,10] or near the corner of jointed dissimilar materials [9,11–13], sinusoidal terms (Tables 1 and 2) appear. On the other hand, examining a plane domain [18] of a composite thick beam subjected to external loads and with various boundary conditions, it is observed that in the case of nonzero eigensolution the stress and displacement fields contain sinusoidal terms given as functions of the cross section's height, when the characteristic polynomial of the governing differential equation has complex roots (Ref. [18], relation (30c)). Moreover in the case of a crack between dissimilar media [7] or layered materials [8] or of the interfacial fracture of sandwich beams [10], it has been shown that the stress field, as well as the crack flank displacements at a distance r from the crack's tip, oscillates with a geometrically increasing period, as r increases. Hence in the expressions obtained in Ref. [7] (relations (5.2)–(5.5)), Ref. [8] (relation (2.27)), and Ref. [10] (relation (5)) appear sinusoidal terms of the form given in Tables 1 and 2. It is also remarkable that these sinusoidal terms under the exponential form $\exp(i\omega x)$ (where x is a Cartesian coordinate and ω is the circular frequency) are also used in wave propagation problems [19].

The variable-separable solutions of the forms $f(\theta)\cos(m \ln r)$ and $f(\theta)\sin(m \ln r)$ (Tables 1 and 2) express terms in the stress and displacement field relations, varying in a particular range of values with a period increasing geometrically according to the distance from the singular point. In particular, the interference of these expressions in the stress and displacement fields concerns terms oscillating strongly near the singular point and widely as we move away from it (e.g., for the crack tip of an interface crack, for the apex of a composite wedge, etc).

The additional separated-variable solutions of the biharmonic equation in polar coordinates improve the accuracy of the solution of the biharmonic equation in polar coordinates because they include the case of complex eigenvalues. From this study occurs that the already tabulated Michell solutions do not consist a complete set of solutions. This happens because the proposed additional terms cannot be analyzed in series of the already tabulated ones [4]. Indeed expanding in power series $\cos(m \ln r)$ occurs [22]

$$\cos(m \ln r) = \sum_{l=0}^{\infty} (-1)^l \frac{m^{2l}(\ln r)^{2l}}{(2l)!}, \quad e^{m\theta} = \sum_{k=0}^{\infty} \frac{(m\theta)^k}{k!} \quad (72)$$

The term $e^{m\theta}\cos(m \ln r)$ using relation (72) is written as

$$e^{m\theta}\cos(m \ln r) = \left(\sum_{k=0}^{\infty} \frac{(m\theta)^k}{k!}\right)\sum_{l=0}^{\infty} (-1)^l \frac{m^{2l}(\ln r)^{2l}}{(2l)!} \quad (73)$$

From relation (73) results that the term $e^{m\theta}\cos(m \ln r)$ cannot be obtained by the superposition of the already tabulated Michell's solution [4]. The same observation is also valid for the terms $e^{-m\theta}\cos(m \ln r)$, $e^{m\theta}\sin(m \ln r)$, and $e^{-m\theta}\sin(m \ln r)$. Thus, the additional separated-variable solutions $\bar{\varphi}_{-m}(r,\theta)$ (relation (53)) cannot be written in terms of the tabulated Michell's solution.

On the other hand considering that $\cos(m \ln r)=t$ and using its expansions in power series [22], it may be written that

$$m \ln r = \arccos(t) = \frac{\pi}{2} - \arcsin(t) = \frac{\pi}{2} - \sum_{l=0}^{\infty} \frac{(2l)!\, t^{2l+1}}{2^{2l}(l!)^2(2l+1)} \quad (74)$$

and

$$\ln r = \sum_{l=0}^{\infty} (-1)^l \frac{(r-1)^{l+1}}{l+1} \quad (75)$$

From relations (74) and (75) an infinite power series functional relation between the radial coordinate r and the term $t=\cos(m \ln r)$ is obtained as

$$\sum_{l=0}^{\infty} \left[(-1)^l \frac{(r-1)^{l+1}}{l+1} + \left(\frac{1}{m}\right)\frac{(2l)!\, t^{2l+1}}{2^{2l}(l!)^2(2l+1)}\right] = \frac{\pi}{2m},$$

$$t = \cos(m \ln r) \quad (76)$$

The functional relation (76) cannot result as a closed form solution of the form $t=t(r)$. The same observation is also valid between the radial coordinate r and the term $s=\sin(m \ln r)$. Thus the proposed additional separated-variable solutions of the biharmonic equation in polar coordinates complete the already tabulated Michell's solution [1,4].

The solution of a differential equation includes unknown coefficients determined from the boundary conditions. The selection of the real constants b, b_1, and b_2 (relations (10) and (24)) in the proposed investigation process does not influence the resulted solutions. For this reason the real constants b, b_1, and b_2, which are included in this investigation, were taken as functions of the physical indices m $(0 \le m \le n)$ of the partial solutions (relation (54)). Even if the investigation of the biharmonic equation is made under a real solution process, the resulted partial solutions coincide with those derived from a complex form solution such as $z^{\delta}=(re^{i\theta})^{a+i\beta}$.

From the application (Sec. 4) it is also proved that the stress function of Williams in the case of a two-dimensional, n-material wedge or junction [13] is related with the additional separated-variable solutions of the biharmonic equation in polar coordinates. Hence it is verified that the existence of the proposed additional terms expands the use of Michell's solution in plane elasticity problems.

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Appendix

Let a homogenous ordinary differential equation (ODE) of the form

$$\alpha_0 y^{(n)} + \alpha_1 y^{(n-1)} + \alpha_2 y^{(n-2)} + \ldots + \alpha_{n-1} y' + a_n y = 0 \qquad \text{(A1)}$$

The characteristic polynomial of Eq. (A1) is

$$\alpha_0 \rho^n + \alpha_1 \rho^{n-1} + \alpha_2 \rho^{n-2} + \ldots + \alpha_{n-1} \rho + a_n = 0 \qquad \text{(A2)}$$

The polynomial (A2) possesses the following roots:

(i) the real roots ρ_j, $j=1,2,\ldots,k$ with degree of multiplicity ν_j

(ii) the complex conjugate roots $r_\ell \pm i\zeta_\ell$, $\ell=1,2,\ldots,\mu$ with degree of multiplicity ξ_ℓ

The general form of the solution of the homogenous ODE (A1) is

$$y = \sum_{j=1}^{k} \sum_{\nu_j=1}^{\nu_j} C_{j\nu_j} x^{\nu_j - 1} e^{\rho_j x} + \sum_{\ell=1}^{\mu} \sum_{\xi_\ell=1}^{\xi_\ell} (A_{\ell\xi_\ell} \cos \zeta_\ell x + B_{\ell\xi_\ell} \sin \zeta_\ell x) x^{\xi_\ell - 1} e^{r_\ell x} \qquad \text{(A3)}$$

where $C_{j\nu_j}$, $A_{\ell\xi_\ell}$, and $B_{\ell\xi_\ell}$ constants are to be determined from the initial and the boundary conditions of the problem.

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